113 Class Problems: Irreducible Elements and Unique Factorization Domains

1. Is the polynomial $2x^2 - 4$ irreducible in $\mathbb{C}[x]$? How about in $\mathbb{R}[x]$, $\mathbb{Q}[x]$ or $\mathbb{Z}[x]$?

   Solutions:

$$\mathbb{C}[x]: \quad 2x^2-4 = 2\overset{\notin\, \mathbb{C}[x]^*}{(x+\sqrt{2})}\overset{\notin\, \mathbb{C}[x]^*}{(x-\sqrt{2})} \implies \text{Reducible in } \mathbb{C}[x]$$

$$\mathbb{R}[x]: \quad 2x^2-4 = 2\overset{\notin\, \mathbb{R}[x]^*}{(x+\sqrt{2})}\overset{\notin\, \mathbb{R}[x]^*}{(x-\sqrt{2})} \implies \text{Reducible in } \mathbb{R}[x]$$

$$\mathbb{Q}[x]: \quad 2x^2-4 = (ax+b)(cx+d), \quad a,b,c,d \in \mathbb{Q}$$

$$\implies \left\{\frac{-b}{a}, \frac{-d}{c}\right\} = \{\sqrt{2}, -\sqrt{2}\} \implies \sqrt{2} \in \mathbb{Q} \quad \underline{\text{contradiction}}$$

$$\implies \text{Irreducible in } \mathbb{Q}[x]$$

$$\mathbb{Z}[x] = \underset{\notin\, \mathbb{Z}[x]^*}{2}\,\underset{\notin\, \mathbb{Z}[x]^*}{(x^2-2)} \implies \text{Reducible in } \mathbb{Z}[x]$$

2. If a polynomial $f(x) \in \mathbb{Q}[x]$ has no roots in $\mathbb{Q}$ must it be irreducible in $\mathbb{Q}[x]$?

   Solutions:

No! $(x^2+1)(x^2+1)$ is reducible in $\mathbb{Q}[x]$ but has no roots in $\mathbb{Q}$.

3. Consider the subring $\mathbb{Z}[\sqrt{-5}] \subset \mathbb{C}$

(a) Prove that $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$

(b) If $a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ is non-zero, what is the minimum possible value of $|a + b\sqrt{-5}|^2$, the square of the absolute value?

(c) Using part (b) determine $\mathbb{Z}[\sqrt{-5}]^*$, the units in $\mathbb{Z}[\sqrt{-5}]$.

(d) Prove that $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are non-associated elements of $\mathbb{Z}[\sqrt{-5}]$.

(e) Prove that $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.

(f) Prove that $\mathbb{Z}[\sqrt{-5}]$ is **not** a UFD.

Solutions:

a)

$(\sqrt{-5})^2 = -5 \in \mathbb{Z} \Rightarrow \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$

b) $|a + b\sqrt{-5}|^2 = a^2 + 5b^2$, $a, b \in \mathbb{Z}$

$\Rightarrow |a + b\sqrt{-5}|^2 \geq 1$ if $a, b \in \mathbb{Z}$. Min value is when $a = \pm 1$ and $b = 0$.

c) $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}] \Rightarrow |\alpha|^2, |\beta|^2 \geq 1$

$\alpha\beta = 1 \Rightarrow |\alpha|^2 |\beta|^2 = 1 \Rightarrow |\alpha|^2 = |\beta|^2 = 1 \Rightarrow \alpha = \pm 1$

$\Rightarrow \mathbb{Z}[\sqrt{-5}]^*$

d) $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are pairwise non-associated as

$\mathbb{Z}[\sqrt{-5}]^* = \{\pm 1\}$ and non is a negative of another

e) $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$

$2 = \alpha\beta \Rightarrow 2^2 = |\alpha|^2 |\beta|^2 \Rightarrow |\alpha|^2 = 1, 2, 4$

$a^2 + 5b^2 = 2$ has no integer solutions $\Rightarrow |\alpha|^2 = 1$ or 4

$|\alpha|^2 = 1 \Rightarrow \alpha \in \mathbb{Z}[\sqrt{-5}]^*$, $|\alpha|^2 = 4 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta \in \mathbb{Z}[\sqrt{-5}]^*$

$\Rightarrow 2$ irreducible.

Same logic shows $3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are all irreducible

f) $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ ← two non-associated irreducible factorizations

$\Rightarrow \mathbb{Z}[\sqrt{-5}]$ <u>not</u> a UFD